Programming a quantum computer – Day 1
Exercises

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1 Theory questions

1. **Basic quantum states**

   Let $|\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle \otimes (|0\rangle + |1\rangle)$ and $|\phi\rangle = \frac{i}{\sqrt{2}} |01\rangle + \frac{1}{\sqrt{2}} |10\rangle$. Compute:

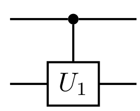
   (a) $\langle\psi|, \quad \langle\phi|$

   (b) $\langle\phi|\psi\rangle$

   (c) $|\psi\rangle\langle\phi|, \quad |\psi\rangle\langle\psi|$

2. **Unitary operators**

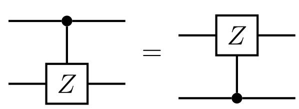
   (a) Let U_1, U_2 be unitary operators. Show that

   $U_1 U_2, \qquad U_1 \otimes U_2,$ 

   ,

   where drawn gate denotes a controlled U_1, are all unitary operators.

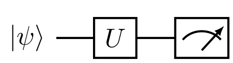
   (b) Let $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ show that

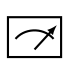
   

   (c) (Bonus) How would you implement an X gate if you could only apply Z and H gates to a qubit?

3. **Simulating measurements in other bases**

   Suppose we want to measure a qubit $|\psi\rangle$ in a basis $\{|v_0\rangle, |v_1\rangle\}$. Find the unitary U such that the circuit

   implements this, where  denotes measurement in the computational basis. I.e., the probability of obtaining outcome k is

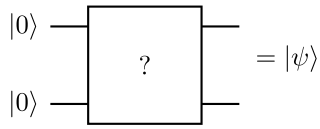
   $$P(k) = \langle\psi|v_k\rangle\langle v_k|\psi\rangle .$$

4. **Creating an entangled state**

   Consider the two-qubit state $|\psi\rangle = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle$.

(a) Show that $|\psi\rangle$ is entangled.

(b) Design a two-qubit circuit that prepares the state $|\psi\rangle$



(Hint: you only need to use standard gates.)

(c) (Bonus) What about the states $|\psi(\theta)\rangle = \cos(\theta)\,|00\rangle + \sin(\theta)\,|11\rangle$?

5. **Random number generator**

(a) Design and implement a one-qubit circuit that generates uniformly random bits.

(b) Design and implement a one-qubit circuit that generates a biased random bit, i.e., $\mathbb{P}(0) = p$ and $\mathbb{P}(1) = 1 - p$ for $p \in [0, 1]$.

6. **Global phase is not physical**

Prove that global phase it not observable. I.e., if $|\psi\rangle = e^{it}\,|\phi\rangle$ then they give the same statistics for all quantum circuits.

7. **Distinguishing quantum states**

Consider the following scenario. You will receive a qubit in either the state $|\psi_0\rangle$ or $|\psi_1\rangle$ (chosen with uniform probability 1/2). Your task is to determine if you received state 0 or state 1.

Suppose you measured in the orthonormal basis $\{|v_0\rangle\,, |v_1\rangle\}$. Then the probability you guess correctly is

$$\mathbb{P}(\text{Succeed}) = \frac{1}{2}\mathbb{P}(\text{ Measure 0} \mid \text{Received state } |\psi_0\rangle) + \frac{1}{2}\mathbb{P}(\text{ Measure 1} \mid \text{Received state } |\psi_1\rangle)$$

which is given by

$$\mathbb{P}(\text{Succeed}) = \frac{1}{2}\,\langle \psi_0 \,|\, v_0\rangle\langle v_0 \,|\, \psi_0\rangle + \frac{1}{2}\,\langle \psi_1 \,|\, v_1\rangle\langle v_1 \,|\, \psi_1\rangle$$

(a) If $|\psi_0\rangle = |0\rangle$ and $|\psi_1\rangle = |1\rangle$ find a measurement so that $\mathbb{P}(\text{Succeed}) = 1$.

(b) Show that

$$\mathbb{P}(\text{Succeed}) = 1 \iff \langle \psi_0 \,|\, \psi_1\rangle = 0$$

(c) Suppose $|\psi_0\rangle = |0\rangle$ and $|\psi_1\rangle = |+\rangle$. Find $\theta \in [0, 2\pi)$ such that the measurement

$$\{\cos(\theta/2)\,|0\rangle + \sin(\theta/2)\,|1\rangle\,,\ \sin(\theta/2)\,|0\rangle - \cos(\theta/2)\,|1\rangle\}$$

maximizes the probability of guessing correctly $\mathbb{P}(\text{Succeed})$.

(d) (Bonus) Would applying a unitary before measuring help to distinguish the two states?

8. **Swap gate**

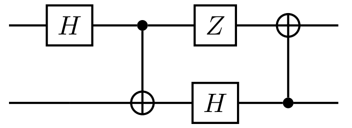
Write down the quantum gate U that swaps 2 qubits. I.e., for qubit states $|\psi\rangle$ and $|\phi\rangle$ we have

$$U\,|\psi\rangle\,|\phi\rangle = |\phi\rangle\,|\psi\rangle\ .$$

Can you generate this gate using only controlled not gates?

9. **A circuit**

(a) Write down the unitary matrix U corresponding to this circuit (written with respect to the computational basis)



(b) Write down the circuit corresponding to the inverse operation U^{-1}.

(c) If the circuit is applied to the initial state $|0\rangle|0\rangle$ and we measure each qubit in the computational basis then what is the distribution of outcomes?

2 Practical exercises

For today's practical you should create an account to access the IBM machines and start to experiment on them either through the graphical composer or by writing qiskit scripts and start running your first quantum program!

You are free to explore as you wish but I also include here some suggestions to get you started:

- Try implementing some of the things discussed in the questions above or in the lecture notes, e.g., generate some (Truly!) random bits or explore your ability to distinguish different states.
- Do the statistics of your programs match up with the theoretical predictions? Why?
- We've seen a quantum not gate $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is relatively simple, $X|0\rangle = |1\rangle$ and $X|1\rangle = |0\rangle$. But what about other standard 'classical' logic gates. How would we for instance implement an XOR gate, or an AND gate, or an OR gate? (Note: you may need additional qubits.) Design some circuits to compute these logic gates for bits encoded as $|0\rangle$ and $|1\rangle$. Try to run them on the IBM machines.