New Rényi divergence families defined via convex optimization and their applications

Peter Brown, Hamza Fawzi and Omar Fawzi

Feb 01, 2021

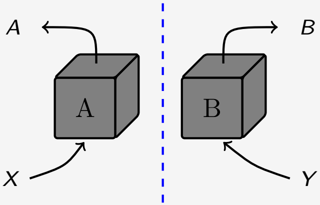
Part 1

The *iterated mean* divergences and their application to device-independent cryptography

Based on *Brown, P., Fawzi, H. and Fawzi, O., Computing conditional entropies for quantum correlations, Nat Commun 12, 575 (2021), arXiv:2007.12575*.

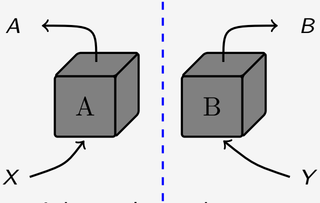
Motivation I

Bell-nonlocality



Motivation I

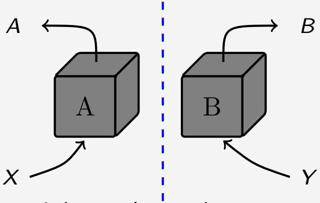
Bell-nonlocality



- Nonlocal correlations are inherently random.

Motivation I

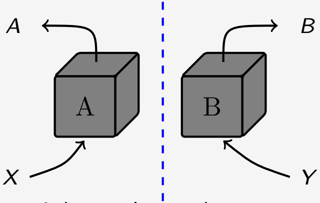
Bell-nonlocality



- Nonlocal correlations are inherently random.
- Foundation for randomness expansion / key-distribution protocols!

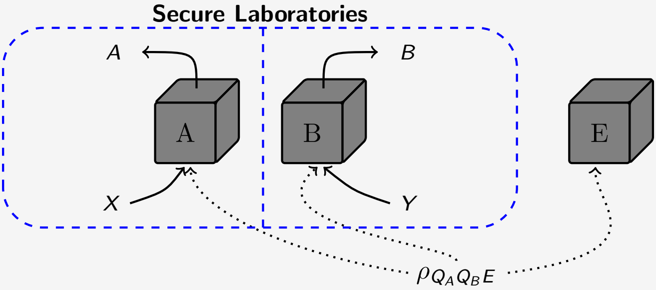
Motivation I

Bell-nonlocality

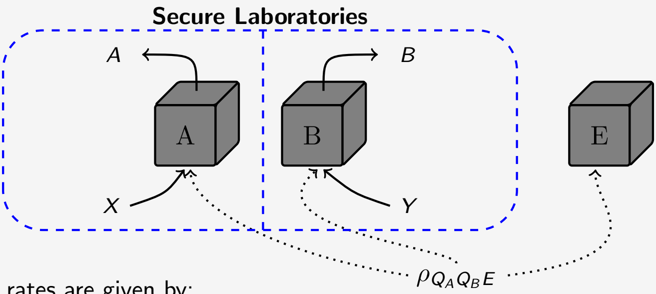


- Nonlocal correlations are inherently random.
- Foundation for randomness expansion / key-distribution protocols!
- Security and analysis relies on being able to calculate the *rates* of such protocols (bits per round).

Randomness generated per round



Randomness generated per round



Asymptotic rates are given by:

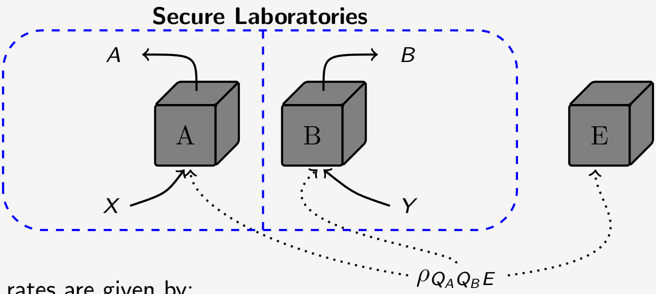
- **Randomness expansion**

$$H(AB|X = x^*, Y = y^*, E)$$

- **QKD**

$$H(A|X = x^*, E) - H(A|X = x^*, Y = y^*, B)$$

Randomness generated per round



Asymptotic rates are given by:

- **Randomness expansion**

$$H(AB|X = x^*, Y = y^*, E)$$

- **QKD**

$$H(A|X = x^*, E) - H(A|X = x^*, Y = y^*, B)$$

Want device-independent lower bounds

Example

We want lower bounds on

$$\begin{aligned} \inf \quad & H(A|X = x^*, E) \\ \text{s.t.} \quad & \sum_{abxy} c^i_{abxy}\, p(ab|xy) = w^i \end{aligned}$$

where the infimum is over all finite dimensional **states** $\rho_{Q_A Q_B E}$, **POVMs** $\{\{M_{a|x}\}_a\}_x, \{\{N_{b|y}\}_b\}_y$ and joint **Hilbert spaces** $Q_A \otimes Q_B \otimes E$.

Example

We want lower bounds on

Difficult to solve
nonconvex / unbounded dimension

$$\begin{aligned} \inf \quad & H(A|X=x^*,E) \\ \text{s.t.} \quad & \sum_{abxy} c^i_{abxy}\, p(ab|xy) = w^i \end{aligned}$$

where the infimum is over all finite dimensional **states** $\rho_{Q_A Q_B E}$, **POVMs** $\{\{M_{a|x}\}_a\}_x, \{\{N_{b|y}\}_b\}_y$ and joint **Hilbert spaces** $Q_A \otimes Q_B \otimes E$.

Example

We want lower bounds on

Difficult to solve
nonconvex / unbounded dimension

$$\begin{aligned} \inf \quad & H(A|X = x^*, E) \\ \text{s.t.} \quad & \sum_{abxy} c^i_{abxy} p(ab|xy) = w^i \end{aligned}$$

where the infimum is over all finite dimensional **states** $\rho_{Q_A Q_B E}$, **POVMs** $\{\{M_{a|x}\}_a\}_x, \{\{N_{b|y}\}_b\}_y$ and joint **Hilbert spaces** $Q_A \otimes Q_B \otimes E$.

Known approaches

- Analytical bounds [PAB$^+$09] – tight bounds / restricted scope
- Numerical bounds on $H_{\min}$ – easy to compute / poor bounds
- Recent work [TSG$^+$19] – good bounds / computationally intensive

Example

We want lower bounds on

Difficult to solve
nonconvex / unbounded dimension

$$\begin{aligned} \inf \quad & H(A|X=x^*, E) \\ \text{s.t.} \quad & \sum_{abxy} c^i_{abxy}\, p(ab|xy) = w^i \end{aligned}$$

where the infimum is over all finite dimensional **states** $\rho_{Q_A Q_B E}$, **POVMs** $\{\{M_{a|x}\}_a\}_x, \{\{N_{b|y}\}_b\}_y$ and joint **Hilbert spaces** $Q_A \otimes Q_B \otimes E$.

Known approaches

- Analytical bounds [PAB$^+$09] – tight bounds / restricted scope
- Numerical bounds on $H_{\min}$ – easy to compute / poor bounds
- Recent work [TSG$^+$19] – good bounds / computationally intensive

Our approach

- Define new conditional entropies that are easy to bound device-independently and lower bound $H(A|E)$.

The IM divergences

Entropies are special cases of divergences

$$\mathbb{H}^{\uparrow}(A|B)_{\rho} := \sup_{\sigma_B} -\mathbb{D}(\rho_{AB} \| I \otimes \sigma_B)$$

or

$$\mathbb{H}^{\downarrow}(A|B)_{\rho} := -\mathbb{D}(\rho_{AB} \| I \otimes \rho_B).$$

The IM divergences

Entropies are special cases of divergences

$$\mathbb{H}^{\uparrow}(A|B)_\rho := \sup_{\sigma_B} -\mathbb{D}(\rho_{AB} \| I \otimes \sigma_B)$$

or

$$\mathbb{H}^{\downarrow}(A|B)_\rho := -\mathbb{D}(\rho_{AB} \| I \otimes \rho_B).$$

We define our conditional entropies via a divergence

The IM divergences

Definition (Iterated mean divergences)

Let $\alpha_k = 2^k/(2^k-1)$ for $k=1,2,\ldots$. Then the **iterated mean** divergences are defined as

$$D_{(\alpha_k)}(\rho\|\sigma) := \frac{1}{\alpha_k - 1}\log Q_{(\alpha_k)}(\rho\|\sigma)\ , \tag{1}$$

with

$$\begin{aligned} Q_{(\alpha_k)}(\rho\|\sigma) := \max_{V_1,\ldots,V_k,Z}\ & \alpha_k \mathrm{Tr}\left[\rho\frac{(V_1+V_1^*)}{2}\right] - (\alpha_k-1)\mathrm{Tr}\left[\sigma Z\right] \\ \text{s.t.}\quad & V_1+V_1^* \geq 0 \\ & \begin{pmatrix} I & V_1 \\ V_1^* & \frac{(V_2+V_2^*)}{2}\end{pmatrix} \geq 0\ \begin{pmatrix} I & V_2 \\ V_2^* & \frac{(V_3+V_3^*)}{2}\end{pmatrix} \geq 0 \cdots \begin{pmatrix} I & V_k \\ V_k^* & Z\end{pmatrix} \geq 0, \end{aligned} \tag{2}$$

where the optimization varies over $V_1,\ldots,V_k, Z \in \mathscr{L}(\mathcal{H})$.

The IM divergences

Discrete family – $(2, \frac{4}{3}, \frac{8}{7}, \frac{16}{15}, \ldots$

Definition (Iterated mean divergences)

Let $\alpha_k = 2^k/(2^k - 1)$ for $k = 1, 2, \ldots$. Then the **iterated mean** divergences are defined as

$$D_{(\alpha_k)}(\rho\|\sigma) := \frac{1}{\alpha_k - 1} \log Q_{(\alpha_k)}(\rho\|\sigma) \ , \tag{1}$$

with

$$\begin{aligned} Q_{(\alpha_k)}(\rho\|\sigma) := \max_{V_1,\ldots,V_k,Z} \ & \alpha_k \mathrm{Tr}\left[\rho \frac{(V_1 + V_1^*)}{2}\right] - (\alpha_k - 1)\mathrm{Tr}\left[\sigma Z\right] \\ \text{s.t.} \quad & V_1 + V_1^* \geq 0 \\ & \begin{pmatrix} I & V_1 \\ V_1^* & \frac{(V_2+V_2^*)}{2} \end{pmatrix} \geq 0 \ \begin{pmatrix} I & V_2 \\ V_2^* & \frac{(V_3+V_3^*)}{2} \end{pmatrix} \geq 0 \ \cdots \ \begin{pmatrix} I & V_k \\ V_k^* & Z \end{pmatrix} \geq 0, \end{aligned} \tag{2}$$

where the optimization varies over $V_1, \ldots, V_k, Z \in \mathscr{L}(\mathcal{H})$.

The IM divergences

Discrete family – $(2, \frac{4}{3}, \frac{8}{7}, \frac{16}{15}, \ldots$

Definition (Iterated mean divergences)

Let $\alpha_k = 2^k/(2^k - 1)$ for $k = 1, 2, \ldots$. Then the **iterated mean** divergences are defined as

$$D_{(\alpha_k)}(\rho\|\sigma) := \frac{1}{\alpha_k - 1} \log Q_{(\alpha_k)}(\rho\|\sigma) \;, \tag{1}$$

with

$$\begin{aligned} Q_{(\alpha_k)}(\rho\|\sigma) := \max_{V_1,\ldots,V_k,Z} \quad & \alpha_k \mathrm{Tr}\left[\rho \frac{(V_1 + V_1^*)}{2}\right] - (\alpha_k - 1)\mathrm{Tr}\left[\sigma Z\right] \\ \text{s.t.} \quad & V_1 + V_1^* \geq 0 \\ & \begin{pmatrix} I & V_1 \\ V_1^* & \frac{(V_2+V_2^*)}{2} \end{pmatrix} \geq 0 \; \begin{pmatrix} I & V_2 \\ V_2^* & \frac{(V_3+V_3^*)}{2} \end{pmatrix} \geq 0 \; \cdots \; \begin{pmatrix} I & V_k \\ V_k^* & Z \end{pmatrix} \geq 0, \end{aligned} \tag{2}$$

Defined via SDP

where the optimization varies over $V_1, \ldots, V_k, Z \in \mathscr{L}(\mathcal{H})$.

The IM divergences

Discrete family – $(2, \frac{4}{3}, \frac{8}{7}, \frac{16}{15}, \ldots$

Definition (Iterated mean divergences)

Let $\alpha_k = 2^k/(2^k - 1)$ for $k = 1, 2, \ldots$. Then the **iterated mean** divergences are defined as

$$D_{(\alpha_k)}(\rho\|\sigma) := \frac{1}{\alpha_k - 1} \log Q_{(\alpha_k)}(\rho\|\sigma) \,, \tag{1}$$

with

Linear in ρ and σ

$$\begin{aligned} Q_{(\alpha_k)}(\rho\|\sigma) := \max_{V_1,\ldots,V_k,Z} \quad & \alpha_k \mathrm{Tr}\left[\rho \frac{(V_1 + V_1^*)}{2}\right] - (\alpha_k - 1)\mathrm{Tr}\left[\sigma Z\right] \\ \text{s.t.} \quad & V_1 + V_1^* \geq 0 \\ & \begin{pmatrix} I & V_1 \\ V_1^* & \frac{(V_2+V_2^*)}{2} \end{pmatrix} \geq 0 \begin{pmatrix} I & V_2 \\ V_2^* & \frac{(V_3+V_3^*)}{2} \end{pmatrix} \geq 0 \cdots \begin{pmatrix} I & V_k \\ V_k^* & Z \end{pmatrix} \geq 0, \end{aligned} \tag{2}$$

Defined via SDP

where the optimization varies over $V_1, \ldots, V_k, Z \in \mathscr{L}(\mathcal{H})$.

The IM divergences

Discrete family – $(2, \frac{4}{3}, \frac{8}{7}, \frac{16}{15}, \ldots$

Definition (Iterated mean divergences)

Let $\alpha_k = 2^k/(2^k - 1)$ for $k = 1, 2, \ldots$. Then the **iterated mean** divergences are defined as

$$D_{(\alpha_k)}(\rho\|\sigma) := \frac{1}{\alpha_k - 1} \log Q_{(\alpha_k)}(\rho\|\sigma) \ , \tag{1}$$

with

Linear in ρ and σ

$$\begin{aligned} Q_{(\alpha_k)}(\rho\|\sigma) := \max_{V_1,\ldots,V_k,Z} \ & \alpha_k \mathrm{Tr}\left[\rho \frac{(V_1 + V_1^*)}{2}\right] - (\alpha_k - 1)\mathrm{Tr}\left[\sigma Z\right] \\ \text{s.t.} \quad & V_1 + V_1^* \geq 0 \\ & \begin{pmatrix} I & V_1 \\ V_1^* & \frac{(V_2+V_2^*)}{2} \end{pmatrix} \geq 0 \ \begin{pmatrix} I & V_2 \\ V_2^* & \frac{(V_3+V_3^*)}{2} \end{pmatrix} \geq 0 \ \cdots \ \begin{pmatrix} I & V_k \\ V_k^* & Z \end{pmatrix} \geq 0, \end{aligned} \tag{2}$$

Defined via SDP

where the optimization varies over $V_1, \ldots, V_k, Z \in \mathscr{L}(\mathcal{H})$.

Structure independent of the dimension!

IM divergence properties

- Satisfies data processing

$$D_{(\alpha_k)}(\mathcal{E}(\rho)\|\mathcal{E}(\sigma)) \leq D_{(\alpha_k)}(\rho\|\sigma) \qquad \forall \text{ channels } \mathcal{E}.$$

IM divergence properties

- Satisfies data processing

$$D_{(\alpha_k)}(\mathcal{E}(\rho)\|\mathcal{E}(\sigma)) \leq D_{(\alpha_k)}(\rho\|\sigma) \qquad \forall \text{ channels } \mathcal{E}.$$

- Lies between geometric and sandwiched

$$\widetilde{D}_{\alpha_k}(\rho\|\sigma) \leq D_{(\alpha_k)}(\rho\|\sigma) \leq \widehat{D}_{\alpha_k}(\rho\|\sigma)$$

Conditional entropies will lower bound H

IM divergence properties

- Satisfies data processing

$$D_{(\alpha_k)}(\mathcal{E}(\rho)\|\mathcal{E}(\sigma)) \leq D_{(\alpha_k)}(\rho\|\sigma) \qquad \forall \text{ channels } \mathcal{E}.$$

- Lies between geometric and sandwiched

$$\widetilde{D}_{\alpha_k}(\rho\|\sigma) \leq D_{(\alpha_k)}(\rho\|\sigma) \leq \widehat{D}_{\alpha_k}(\rho\|\sigma)$$

Conditional entropies will lower bound H

- Decreasing in k

$$D_{(\alpha_k)}(\rho\|\sigma) \leq D_{(\alpha_{k-1})}(\rho\|\sigma)$$

and so for the corresponding conditional entropies

$$H_{(\alpha_k)}(A|B) \geq H_{(\alpha_{k-1})}(A|B)$$

Improving lower bounds on H

IM conditional entropies

Using the IM divergences we can construct a conditional entropy. Given a bipartitie state ρ_{AB} we have

$$H^{\uparrow}_{(\alpha_k)}(A|B)_\rho = \frac{\alpha_k}{1-\alpha_k} \log Q^{\uparrow}_{(\alpha_k)}(\rho) \tag{3}$$

where

$$\begin{aligned} Q^{\uparrow}_{(\alpha_k)}(\rho) = \max_{V_1,\dots,V_k} \ & \mathrm{Tr}\left[\rho \frac{(V_1 + V_1^*)}{2}\right] \\ \text{s.t.} \quad & \mathrm{Tr}_A\left[V_k^* V_k\right] \leq I_B \\ & V_1 + V_1^* \geq 0 \\ & \begin{pmatrix} I & V_1 \\ V_1^* & \frac{(V_2+V_2^*)}{2} \end{pmatrix} \geq 0 \quad \begin{pmatrix} I & V_2 \\ V_2^* & \frac{(V_3+V_3^*)}{2} \end{pmatrix} \geq 0 \quad \cdots \\ & \begin{pmatrix} I & V_{k-1} \\ V_{k-1}^* & \frac{(V_k+V_k^*)}{2} \end{pmatrix} \geq 0 \ . \end{aligned} \tag{4}$$

IM conditional entropies

Using the IM divergences we can construct a conditional entropy. Given a bipartitie state ρ_{AB} we have

$$H^{\uparrow}_{(\alpha_k)}(A|B)_\rho = \frac{\alpha_k}{1-\alpha_k} \log Q^{\uparrow}_{(\alpha_k)}(\rho) \tag{3}$$

where

$$\begin{aligned} Q^{\uparrow}_{(\alpha_k)}(\rho) = \max_{V_1,\ldots,V_k} \quad & \mathrm{Tr}\left[\rho \frac{(V_1+V_1^*)}{2}\right] \\ \text{s.t.} \quad & \mathrm{Tr}_A\left[V_k^* V_k\right] \leq I_B \\ & V_1 + V_1^* \geq 0 \\ & \begin{pmatrix} I & V_1 \\ V_1^* & \frac{(V_2+V_2^*)}{2} \end{pmatrix} \geq 0 \quad \begin{pmatrix} I & V_2 \\ V_2^* & \frac{(V_3+V_3^*)}{2} \end{pmatrix} \geq 0 \quad \cdots \\ & \begin{pmatrix} I & V_{k-1} \\ V_{k-1}^* & \frac{(V_k+V_k^*)}{2} \end{pmatrix} \geq 0 \,. \end{aligned} \tag{4}$$

Form still suitable for DI optimization!

IM conditional entropies II

For example

$$\begin{aligned} H^{\uparrow}_{(2)}(A|B)_\rho = -2 \log \max_{V_1} \ & \mathrm{Tr}\left[\rho \frac{(V_1 + V_1^*)}{2}\right] \\ \text{s.t.} \quad & \mathrm{Tr}_A [V_1^* V_1] \leq I_B \\ & V_1 + V_1^* \geq 0 \end{aligned} \tag{5}$$

Compare with

$$\begin{aligned} H_{\min}(A|B)_\rho = -\log \max \ & \mathrm{Tr}[\rho M] \\ \text{s.t.} \quad & \mathrm{Tr}_A [M] \leq I_B \\ & M \geq 0 \end{aligned} \tag{6}$$

IM conditional entropies II

For example

$$\begin{aligned} H^{\uparrow}_{(2)}(A|B)_\rho = -2 \log \max_{V_1} \ & \mathrm{Tr}\left[\rho \frac{(V_1 + V_1^*)}{2}\right] \\ \text{s.t.} \quad & \mathrm{Tr}_A\left[V_1^* V_1\right] \le I_B \\ & V_1 + V_1^* \ge 0 \end{aligned} \tag{5}$$

Compare with

$$\begin{aligned} H_{\min}(A|B)_\rho = -\log \max \ & \mathrm{Tr}\left[\rho M\right] \\ \text{s.t.} \quad & \mathrm{Tr}_A\left[M\right] \le I_B \\ & M \ge 0 \end{aligned} \tag{6}$$

For DI applications we can rewrite this in terms of the initial entangled state $|\psi\rangle\langle\psi|$ and the POVM operators used by Alice.

Can then be optimized in the Navascués Pironio Acín hierarchy [NPA07].

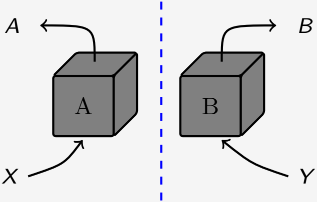
Application: DIRNG/DIQKD setup

- DIRNG – Lower bound

$$H(AB|X = x, Y = y, E)$$

.

- DIQKD – Lower bound



$$H(A|X = x, E) - H(A|B, X = x, Y = y)$$

Application: DIRNG/DIQKD setup

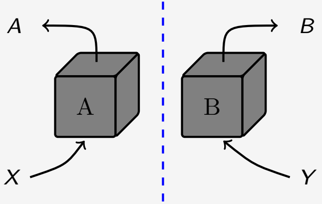
- DIRNG – Lower bound

$$H(AB|X = x, Y = y, E)$$

.

- DIQKD – Lower bound

$$H(A|X = x, E) - H(A|B, X = x, Y = y)$$



- Constrain devices by some full joint probability distribution $p_{AB|XY}$.

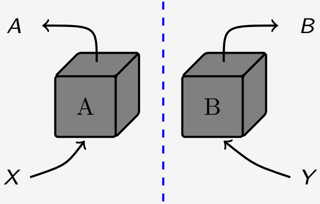
Application: DIRNG/DIQKD setup

- DIRNG – Lower bound

$$H(AB|X = x, Y = y, E)$$

.

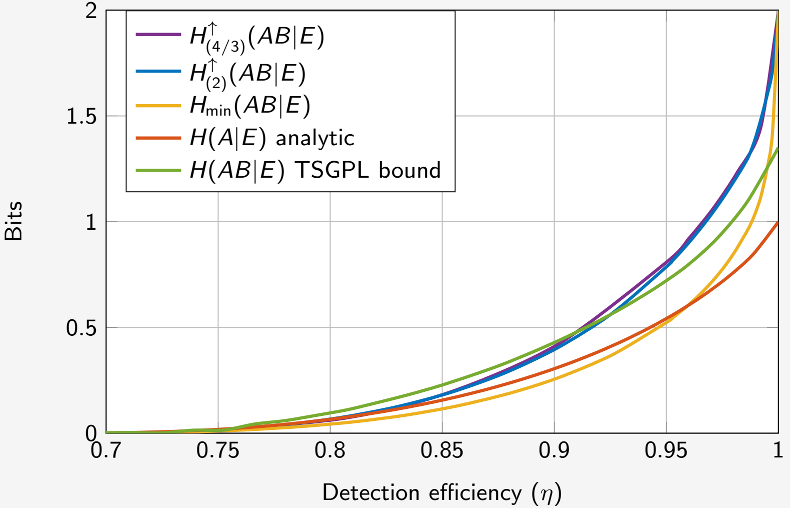
- DIQKD – Lower bound



$$H(A|X = x, E) - H(A|B, X = x, Y = y)$$

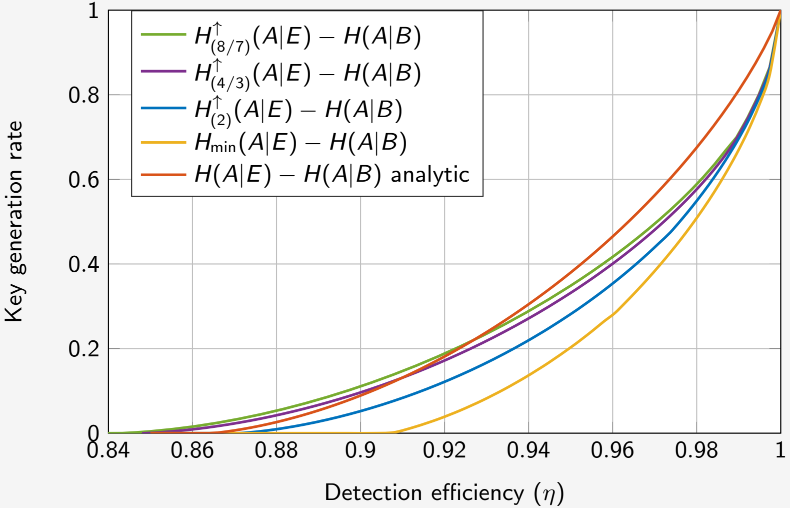
- Constrain devices by some full joint probability distribution $p_{AB|XY}$.
- Assume devices have *detection inefficiencies*. With probability η device measures correctly and with probability $1 - \eta$ device deterministically outputs 0.

Application: DIRNG - full statistics / inefficient detectors

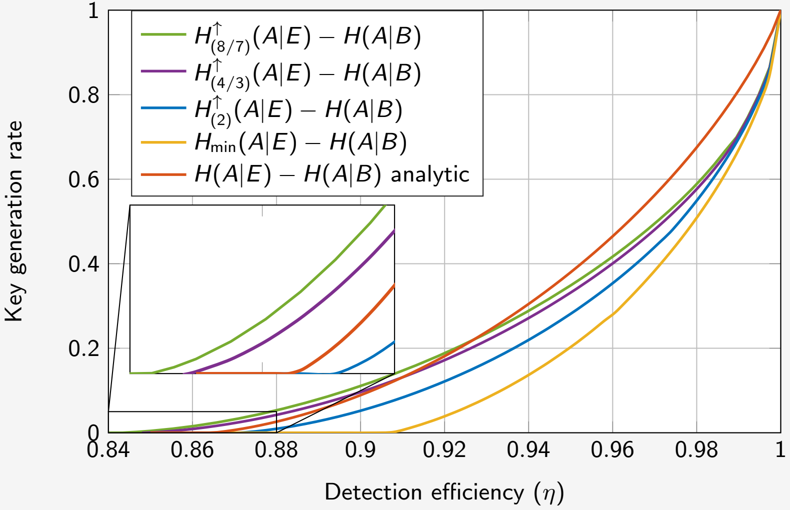


$H(A|E)$ bound from [PAB+09].
TSGPL bound from [TSG+19].

Application: DIQKD - full statistics / inefficient detectors



Application: DIQKD - full statistics / inefficient detectors



Part 2

Divergences defined via convex optimization with applications to quantum Shannon theory

Based on *Fawzi, H. and Fawzi, O., Defining quantum divergences via convex optimization, Quantum, 2021, arXiv:2007.12576*.

Motivation

Divergences are useful quantities in both classical and quantum Shannon theory.

- Can be used to define other important entropic quantities – entropies / mutual information.

Motivation

Divergences are useful quantities in both classical and quantum Shannon theory.

- Can be used to define other important entropic quantities – entropies / mutual information.
- Find direct operational meanings in rates for hypothesis testing – measures of distinguishability.

Motivation

Divergences are useful quantities in both classical and quantum Shannon theory.

- Can be used to define other important entropic quantities – entropies / mutual information.
- Find direct operational meanings in rates for hypothesis testing – measures of distinguishability.
- This work introduces another family of divergences $D_\alpha^{\#}$ which provide new insights for the sandwiched divergences

$$\widetilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \mathrm{Tr}\left[\left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}}\right)^{\alpha}\right].$$

Definition

Given two PSD matrices $A \gg B$ and $\beta \in [0, 1]$, let

$$A \#_\beta B := A^{1/2}(A^{-1/2} B A^{-1/2})^\beta A^{1/2}.$$

Definition

Given two PSD matrices $A \gg B$ and $\beta \in [0,1]$, let

$$A \#_\beta B := A^{1/2}(A^{-1/2} B A^{-1/2})^\beta A^{1/2}.$$

Definition

For $\alpha > 1$ let

$$D_\alpha^\#(\rho\|\sigma) := \frac{1}{\alpha-1} \log Q_\alpha^\#(\rho\|\sigma)$$

where

$$Q_\alpha^\#(\rho\|\sigma) := \min_{A \geq 0} \quad \mathrm{Tr}\,[A] \qquad \text{s.t.} \quad \rho \leq \sigma \#_{1/\alpha} A$$

Definition

Given two PSD matrices $A \gg B$ and $\beta \in [0,1]$, let

$$A \#_\beta B := A^{1/2}(A^{-1/2} B A^{-1/2})^\beta A^{1/2}.$$

Definition

For $\alpha > 1$ let

$$D_\alpha^\#(\rho\|\sigma) := \frac{1}{\alpha - 1} \log Q_\alpha^\#(\rho\|\sigma)$$

where

$$\begin{aligned} Q_\alpha^\#(\rho\|\sigma) := \min_{A \geq 0} \quad & \mathrm{Tr}\,[A] \\ \text{s.t.} \quad & \rho \leq \sigma \#_{1/\alpha} A \end{aligned}$$

SDP when $\alpha \in \mathbb{Q}$

Definition

Given two PSD matrices $A \gg B$ and $\beta \in [0,1]$, let

$$A \#_\beta B := A^{1/2}(A^{-1/2} B A^{-1/2})^\beta A^{1/2}.$$

Definition

For $\alpha > 1$ let

$$D_\alpha^\#(\rho\|\sigma) := \frac{1}{\alpha - 1} \log Q_\alpha^\#(\rho\|\sigma)$$

where

$$Q_\alpha^\#(\rho\|\sigma) := \min_{A \geq 0} \quad \mathrm{Tr}\,[A] \qquad \text{s.t.} \quad \rho \leq \sigma \#_{1/\alpha} A$$

SDP when $\alpha \in \mathbb{Q}$

Same as IM divergence when $\alpha = 2$

Channel divergence

We can also define a corresponding divergence for channels $\mathcal{N}, \mathcal{M} : \mathcal{L}(X') \to \mathcal{L}(Y)$ in the usual way

$$D_\alpha^\#(\mathcal{N}\|\mathcal{M}) = \sup_{\rho_{XX'}} D_\alpha^\#((\mathcal{I} \otimes \mathcal{N})(\rho_{XX'})\|(\mathcal{I} \otimes \mathcal{M})(\rho_{XX'})).$$

Channel divergence

We can also define a corresponding divergence for channels $\mathcal{N}, \mathcal{M} : \mathcal{L}(X') \to \mathcal{L}(Y)$ in the usual way

$$D_\alpha^\#(\mathcal{N}\|\mathcal{M}) = \sup_{\rho_{XX'}} D_\alpha^\#((\mathcal{I} \otimes \mathcal{N})(\rho_{XX'})\|(\mathcal{I} \otimes \mathcal{M})(\rho_{XX'})).$$

For $D_\alpha^\#$ this can be reformulated as a *convex optimization problem*

$$D_\alpha^\#(\mathcal{N}\|\mathcal{M}) = \frac{1}{\alpha - 1} \log Q_\alpha^\#(\mathcal{N}\|\mathcal{M})$$

with

$$\begin{aligned} Q_\alpha^\#(\mathcal{N}\|\mathcal{M}) = \inf_{A_{XY} \geq 0} & \|\mathrm{Tr}_Y [A_{XY}]\|_\infty \\ \text{s.t.} \quad & J_{XY}^{\mathcal{N}} \leq J_{XY}^{\mathcal{M}} \#_{1/\alpha} A_{XY} \end{aligned}$$

Channel divergence

We can also define a corresponding divergence for channels $\mathcal{N}, \mathcal{M} : \mathcal{L}(X') \to \mathcal{L}(Y)$ in the usual way

$$D_\alpha^\#(\mathcal{N}\|\mathcal{M}) = \sup_{\rho_{XX'}} D_\alpha^\#((\mathcal{I}\otimes\mathcal{N})(\rho_{XX'})\|(\mathcal{I}\otimes\mathcal{M})(\rho_{XX'})).$$

For $D_\alpha^\#$ this can be reformulated as a *convex optimization problem*

$$D_\alpha^\#(\mathcal{N}\|\mathcal{M}) = \frac{1}{\alpha-1}\log Q_\alpha^\#(\mathcal{N}\|\mathcal{M})$$

with

$$Q_\alpha^\#(\mathcal{N}\|\mathcal{M}) = \inf_{A_{XY}\geq 0} \|\mathrm{Tr}_Y\,[A_{XY}]\,\|_\infty \quad \text{s.t.} \quad J_{XY}^{\mathcal{N}} \leq J_{XY}^{\mathcal{M}} \#_{1/\alpha} A_{XY}$$

Choi matrices

Properties

- Satisfies data processing

$$D_\alpha^{\#}(\mathcal{E}(\rho)\|\mathcal{E}(\sigma)) \leq D_\alpha^{\#}(\rho\|\sigma) \qquad \forall \text{ channels } \mathcal{E}.$$

Properties

- Satisfies data processing

$$D^{\#}_{\alpha}(\mathcal{E}(\rho)\|\mathcal{E}(\sigma)) \leq D^{\#}_{\alpha}(\rho\|\sigma) \qquad \forall \text{ channels } \mathcal{E}.$$

- Relation to other divergences

$$\widetilde{D}_{\alpha}(\rho\|\sigma) \leq D^{\#}_{\alpha}(\rho\|\sigma) \leq \widehat{D}_{\alpha}(\rho\|\sigma).$$

Properties

- Satisfies data processing

$$D_\alpha^{\#}(\mathcal{E}(\rho)\|\mathcal{E}(\sigma)) \leq D_\alpha^{\#}(\rho\|\sigma) \qquad \forall \text{ channels } \mathcal{E}.$$

- Relation to other divergences

$$\widetilde{D}_\alpha(\rho\|\sigma) \leq D_\alpha^{\#}(\rho\|\sigma) \leq \widehat{D}_\alpha(\rho\|\sigma).$$

- Regularizes to sandwiched divergence

$$\lim_{n\to\infty} \frac{1}{n} D_\alpha^{\#}(\rho^{\otimes n}\|\sigma^{\otimes n}) = \widetilde{D}_\alpha(\rho\|\sigma).$$

Application I: Computing $\lim_{n\to\infty} \frac{1}{n}\widetilde{D}_\alpha(\mathcal{M}^{\otimes n}\|\mathcal{N}^{\otimes n})$

We can use $D_\alpha^{\#}$ to compute

$$\widetilde{D}_\alpha^{\mathrm{reg}}(\mathcal{N}\|\mathcal{M}) := \lim_{n\to\infty} \frac{1}{n}\widetilde{D}_\alpha(\mathcal{M}^{\otimes n}\|\mathcal{N}^{\otimes n})$$

to arbitrary accuracy.

Useful quantity in channel discrimination

Application I: Computing $\lim_{n\to\infty} \frac{1}{n}\widetilde{D}_\alpha(\mathcal{M}^{\otimes n}\|\mathcal{N}^{\otimes n})$

We can use $D^{\#}_\alpha$ to compute

$$\widetilde{D}^{\mathrm{reg}}_\alpha(\mathcal{N}\|\mathcal{M}) := \lim_{n\to\infty} \frac{1}{n}\widetilde{D}_\alpha(\mathcal{M}^{\otimes n}\|\mathcal{N}^{\otimes n})$$

to arbitrary accuracy.

Useful quantity in channel discrimination

Theorem (Informal)

For all $\alpha > 1$ and $m \geq 1$

$$\frac{1}{m} D^{\#}_\alpha(\mathcal{N}^{\otimes m}\|\mathcal{M}^{\otimes m}) - g(m,\alpha) \leq \widetilde{D}^{\mathrm{reg}}_\alpha(\mathcal{N}\|\mathcal{M})$$

and

$$\widetilde{D}^{\mathrm{reg}}_\alpha(\mathcal{N}\|\mathcal{M}) \leq \frac{1}{m} D^{\#}_\alpha(\mathcal{N}^{\otimes m}\|\mathcal{M}^{\otimes m}).$$

Application I: Computing $\lim_{n\to\infty} \frac{1}{n}\widetilde{D}_\alpha(\mathcal{M}^{\otimes n}\|\mathcal{N}^{\otimes n})$

We can use $D^{\#}_\alpha$ to compute

$$\widetilde{D}^{\mathrm{reg}}_\alpha(\mathcal{N}\|\mathcal{M}) := \lim_{n\to\infty} \frac{1}{n}\widetilde{D}_\alpha(\mathcal{M}^{\otimes n}\|\mathcal{N}^{\otimes n})$$

to arbitrary accuracy.

Useful quantity in channel discrimination

Theorem (Informal)

For all $\alpha > 1$ *and* $m \geq 1$

$$\frac{1}{m} D^{\#}_\alpha(\mathcal{N}^{\otimes m}\|\mathcal{M}^{\otimes m}) - g(m,\alpha) \leq \widetilde{D}^{\mathrm{reg}}_\alpha(\mathcal{N}\|\mathcal{M})$$

and

$$\widetilde{D}^{\mathrm{reg}}_\alpha(\mathcal{N}\|\mathcal{M}) \leq \frac{1}{m} D^{\#}_\alpha(\mathcal{N}^{\otimes m}\|\mathcal{M}^{\otimes m}).$$

Can also be used to compute bounds on the relative entropy analogue!

Application II: A new chain rule for $\widetilde{D}_\alpha$

Theorem (Chain rule for $\widetilde{D}_\alpha$)

Let $\alpha > 1$, $\rho, \sigma \geq 0$ *and* $\mathcal{N}, \mathcal{M} : \mathscr{L}(X) \to \mathscr{L}(Y)$ *be quantum channels. Then*

$$\widetilde{D}_\alpha(\mathcal{N}(\rho)\|\mathcal{M}(\sigma)) \leq \widetilde{D}_\alpha^{\mathrm{reg}}(\mathcal{N}\|\mathcal{M}) + \widetilde{D}_\alpha(\rho\|\sigma)$$

Application II: A new chain rule for $\widetilde{D}_\alpha$

Theorem (Chain rule for $\widetilde{D}_\alpha$)

Let $\alpha > 1$, $\rho, \sigma \geq 0$ *and* $\mathcal{N}, \mathcal{M} : \mathscr{L}(X) \to \mathscr{L}(Y)$ *be quantum channels. Then*

$$\widetilde{D}_\alpha(\mathcal{N}(\rho)\|\mathcal{M}(\sigma)) \leq \widetilde{D}_\alpha^{\mathrm{reg}}(\mathcal{N}\|\mathcal{M}) + \widetilde{D}_\alpha(\rho\|\sigma)$$

- Generalization of the DPI

Application II: A new chain rule for $\widetilde{D}_\alpha$

Theorem (Chain rule for $\widetilde{D}_\alpha$)

Let $\alpha > 1$, $\rho, \sigma \geq 0$ *and* $\mathcal{N}, \mathcal{M} : \mathscr{L}(X) \to \mathscr{L}(Y)$ *be quantum channels. Then*

$$\widetilde{D}_\alpha(\mathcal{N}(\rho) \| \mathcal{M}(\sigma)) \leq \widetilde{D}_\alpha^{\mathrm{reg}}(\mathcal{N} \| \mathcal{M}) + \widetilde{D}_\alpha(\rho \| \sigma)$$

- Generalization of the DPI
- Same chain rule already known for the relative entropy [FFRS20]

Application II: A new chain rule for $\widetilde{D}_\alpha$

Theorem (Chain rule for $\widetilde{D}_\alpha$)

Let $\alpha > 1$, $\rho, \sigma \geq 0$ and $\mathcal{N}, \mathcal{M} : \mathscr{L}(X) \to \mathscr{L}(Y)$ be quantum channels. Then

$$\widetilde{D}_\alpha(\mathcal{N}(\rho)\|\mathcal{M}(\sigma)) \leq \widetilde{D}_\alpha^{\mathrm{reg}}(\mathcal{N}\|\mathcal{M}) + \widetilde{D}_\alpha(\rho\|\sigma)$$

- Generalization of the DPI
- Same chain rule already known for the relative entropy [FFRS20]
- Ex: useful for bounding repeated channel applications

$$\widetilde{D}_\alpha(\mathcal{N}^t(\rho)\|\mathcal{M}^t(\sigma)) \leq t\widetilde{D}_\alpha^{\mathrm{reg}}(\mathcal{N}\|\mathcal{M}) + \widetilde{D}_\alpha(\rho\|\sigma)$$

Application III: Channel discrimination

Task: Given black box access to one of the channels $\mathcal{N}, \mathcal{M} : \mathscr{L}(X') \to \mathscr{L}(Y)$, determine if you received $\mathcal{N}$.

Application III: Channel discrimination

Task: Given black box access to one of the channels $\mathcal{N}, \mathcal{M} : \mathscr{L}(X') \to \mathscr{L}(Y)$, determine if you received $\mathcal{N}$.

- Recent work [WBHK20] introduced the *amortized divergence*

$$\mathbb{D}^a(\mathcal{N}\|\mathcal{M}) := \sup_{\rho_{XX'}, \sigma_{XX'} \in \mathscr{D}(XX')} [\mathbb{D}(\mathcal{N}(\rho_{XX'})\|\mathcal{M}(\sigma_{XX'})) - \mathbb{D}(\rho_{XX'}\|\sigma_{XX'})]$$

as a tool for computing rates of this task.

Application III: Channel discrimination

Task: Given black box access to one of the channels $\mathcal{N}, \mathcal{M} : \mathscr{L}(X') \to \mathscr{L}(Y)$, determine if you received $\mathcal{N}$.

- Recent work [WBHK20] introduced the *amortized divergence*

$$\mathbb{D}^a(\mathcal{N}\|\mathcal{M}) := \sup_{\rho_{XX'},\sigma_{XX'}\in\mathscr{D}(XX')} [\mathbb{D}(\mathcal{N}(\rho_{XX'})\|\mathcal{M}(\sigma_{XX'})) - \mathbb{D}(\rho_{XX'}\|\sigma_{XX'})]$$

  as a tool for computing rates of this task.

- Using the chain rule one can prove

$$\widetilde{D}^a_\alpha(\mathcal{N}\|\mathcal{M}) = \widetilde{D}^{\mathrm{reg}}_\alpha(\mathcal{N}\|\mathcal{M}).$$

Application III: Channel discrimination

Task: Given black box access to one of the channels $\mathcal{N}, \mathcal{M} : \mathscr{L}(X') \to \mathscr{L}(Y)$, determine if you received $\mathcal{N}$.

- Recent work [WBHK20] introduced the *amortized divergence*

$$\mathbb{D}^a(\mathcal{N}\|\mathcal{M}) := \sup_{\rho_{XX'}, \sigma_{XX'} \in \mathscr{D}(XX')} [\mathbb{D}(\mathcal{N}(\rho_{XX'})\|\mathcal{M}(\sigma_{XX'})) - \mathbb{D}(\rho_{XX'}\|\sigma_{XX'})]$$

  as a tool for computing rates of this task.
- Using the chain rule one can prove

$$\widetilde{D}^a_\alpha(\mathcal{N}\|\mathcal{M}) = \widetilde{D}^{\mathrm{reg}}_\alpha(\mathcal{N}\|\mathcal{M}).$$

We can compute this!

Application III: Channel discrimination

Task: Given black box access to one of the channels $\mathcal{N}, \mathcal{M} : \mathscr{L}(X') \to \mathscr{L}(Y)$, determine if you received $\mathcal{N}$.

- Recent work [WBHK20] introduced the *amortized divergence*

$$\mathbb{D}^a(\mathcal{N} \| \mathcal{M}) := \sup_{\rho_{XX'}, \sigma_{XX'} \in \mathscr{D}(XX')} [\mathbb{D}(\mathcal{N}(\rho_{XX'}) \| \mathcal{M}(\sigma_{XX'})) - \mathbb{D}(\rho_{XX'} \| \sigma_{XX'})]$$

  as a tool for computing rates of this task.

- Using the chain rule one can prove

$$\widetilde{D}^a_\alpha(\mathcal{N} \| \mathcal{M}) = \widetilde{D}^{\mathrm{reg}}_\alpha(\mathcal{N} \| \mathcal{M}).$$

We can compute this!

- It can also be shown in certain new regimes that adaptive strategies do not help!

Application III: Channel discrimination

Task: Given black box access to one of the channels $\mathcal{N}, \mathcal{M} : \mathscr{L}(X') \to \mathscr{L}(Y)$, determine if you received $\mathcal{N}$.

- Recent work [WBHK20] introduced the *amortized divergence*

$$\mathbb{D}^a(\mathcal{N}\|\mathcal{M}) := \sup_{\rho_{XX'},\sigma_{XX'}\in\mathscr{D}(XX')} [\mathbb{D}(\mathcal{N}(\rho_{XX'})\|\mathcal{M}(\sigma_{XX'})) - \mathbb{D}(\rho_{XX'}\|\sigma_{XX'})]$$

  as a tool for computing rates of this task.

- Using the chain rule one can prove

$$\widetilde{D}^a_\alpha(\mathcal{N}\|\mathcal{M}) = \widetilde{D}^{\mathrm{reg}}_\alpha(\mathcal{N}\|\mathcal{M}).$$

We can compute this!

- It can also be shown in certain new regimes that adaptive strategies do not help!

Strong converse exponent

Further work

- Use to design better DI protocols / apply to different DI tasks. Can we include preprocessing in DIQKD? [HST^{+}20, WAP20]

Further work

- Use to design better DI protocols / apply to different DI tasks. Can we include preprocessing in DIQKD? [HST$^+$20, WAP20]
- Analyze finite round key rates (feasibility of DIQKD).

Further work

- Use to design better DI protocols / apply to different DI tasks. Can we include preprocessing in DIQKD? [HST+20, WAP20]
- Analyze finite round key rates (feasibility of DIQKD).
- Can we make the computations more efficient? (Symmetries/dilations?)

Further work

- Use to design better DI protocols / apply to different DI tasks. Can we include preprocessing in DIQKD? [HST+20, WAP20]
- Analyze finite round key rates (feasibility of DIQKD).
- Can we make the computations more efficient? (Symmetries/dilations?)
- What are the limiting cases as $\alpha \to 1$

$$\lim_{\alpha\to 1} D_{(\alpha)}(\rho\|\sigma) = \,?$$

$$\lim_{\alpha\to 1} D_{\alpha}^{\#}(\rho\|\sigma) = \,?$$

Further work

- Use to design better DI protocols / apply to different DI tasks. Can we include preprocessing in DIQKD? [HST$^+$20, WAP20]
- Analyze finite round key rates (feasibility of DIQKD).
- Can we make the computations more efficient? (Symmetries/dilations?)
- What are the limiting cases as $\alpha \to 1$

$$\lim_{\alpha\to 1} D_{(\alpha)}(\rho\|\sigma) = \,?$$

$$\lim_{\alpha\to 1} D_{\alpha}^{\#}(\rho\|\sigma) = \,?$$

- Other applications to $\widetilde{D}_\alpha$?

Further work

- Use to design better DI protocols / apply to different DI tasks. Can we include preprocessing in DIQKD? [HST+20, WAP20]
- Analyze finite round key rates (feasibility of DIQKD).
- Can we make the computations more efficient? (Symmetries/dilations?)
- What are the limiting cases as $\alpha \to 1$

$$\lim_{\alpha \to 1} D_{(\alpha)}(\rho\|\sigma) = ?$$

$$\lim_{\alpha \to 1} D^{\#}_{\alpha}(\rho\|\sigma) = ?$$

- Other applications to $\widetilde{D}_{\alpha}$?
- Can we construct other families in a similar way?

Bibliography

Kun Fang, Omar Fawzi, Renato Renner, and David Sutter.
Chain rule for the quantum relative entropy.
Phys. Rev. Lett., 124:100501, Mar 2020.

M Ho, P Sekatski, EY-Z Tan, R Renner, J-D Bancal, and N Sangouard.
Noisy preprocessing facilitates a photonic realization of device-independent quantum key distribution.
Physical Review Letters, 124(23):230502, 2020.

Miguel Navascués, Stefano Pironio, and Antonio Acín.
Bounding the set of quantum correlations.
Physical Review Letters, 98:010401, 2007.

Stefano Pironio, Antonio Acín, Nicolas Brunner, Nicolas Gisin, Serge Massar, and Valerio Scarani.
Device-independent quantum key distribution secure against collective attacks.
New Journal of Physics, 11(4):045021, 2009.

Ernest Y-Z Tan, René Schwonnek, Koon Tong Goh, Ignatius William Primaatmaja, and Charles C-W Lim.
Computing secure key rates for quantum key distribution with untrusted devices.
arXiv preprint arXiv:1908.11372, 2019.

Erik Woodhead, Antonio Acín, and Stefano Pironio.
Device-independent quantum key distribution based on asymmetric chsh inequalities.
arXiv preprint arXiv:2007.16146, 2020.

Mark M Wilde, Mario Berta, Christoph Hirche, and Eneet Kaur.
Amortized channel divergence for asymptotic quantum channel discrimination.
Letters in Mathematical Physics, 110(8):2277–2336, 2020.