# Device-independent lower bounds on the conditional von Neumann entropy

Peter Brown, Hamza Fawzi and Omar Fawzi

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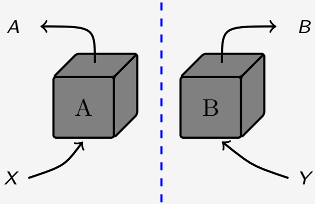
# Motivation I

## Bell-nonlocality



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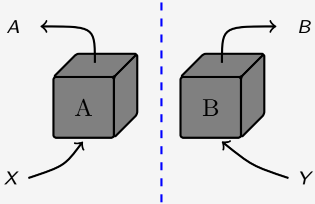


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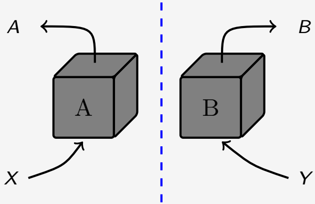


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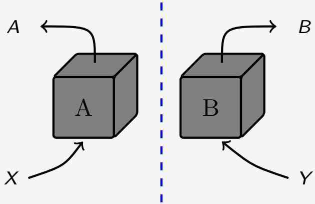


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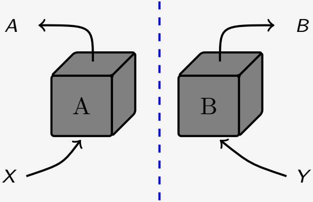


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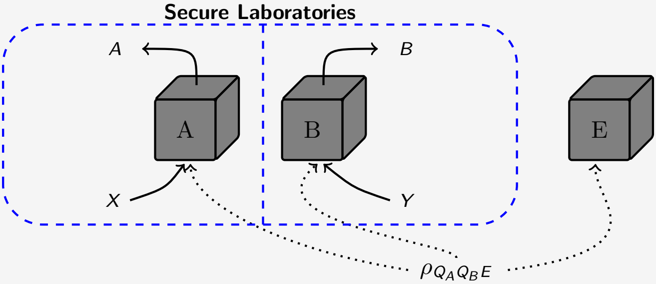
Main task of this work

# Randomness generated per round



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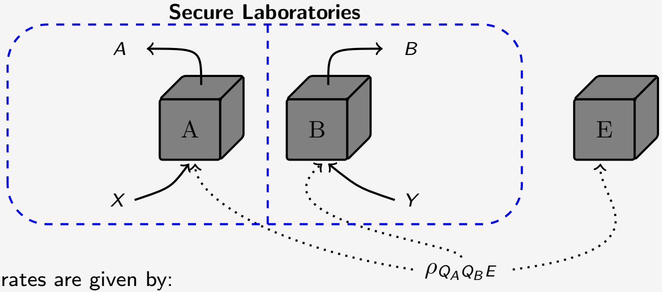


Asymptotic rates are given by:

- **Randomness expansion**

$$H(AB|X = x^*, Y = y^*, E)$$

- **QKD**

$$H(A|X = x^*, E) - H(A|X = x^*, Y = y^*, B)$$

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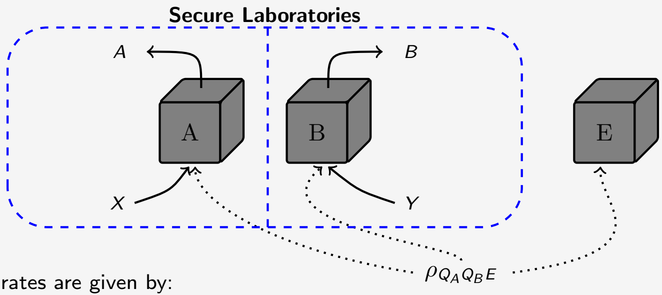


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Want device-independent lower bounds

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- Our new approach
  - Define a sequence

$$H_m(\rho) = \inf_{Z_1,\ldots,Z_m \in B(H)} \mathrm{Tr}\left[\rho\, q(Z_1,\ldots,Z_m)\right] \tag{1}$$

    such that $H_m \leq H$ and $H_m \to H$ as $m \to \infty$.
  - close to optimal / more efficient / wider scope

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with $p_i$ and $q_i$ some polynomials and with the RHS converging as $m \to \infty$.

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Form sufficient for later NPA relaxations

## Derivation overview

1. Gauss-Radau approximation of the logarithm

$$\ln(x) = \int_0^1 \frac{x-1}{t(x-1)+1} \mathrm{d}t \geq \sum_{i=1}^{m} w_i f_{t_i}(x)$$

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## Result

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and RHS converges as $m \to \infty$.

# Lower bound on $H(A|X = x^*, Q_E)$

$H(A|B) = -D(\rho_{AB} \| I_A \otimes \rho_B)$

## Theorem

*The rate* $\inf H(A|X = x^*, Q_E)$ *is never smaller than*

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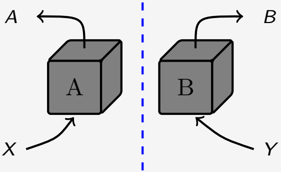
- NPA hierarchy converges as $\|Z\|$ can be bounded.
- Similar results for $H(AB|X = x, Y = y, Q_E)$ or $H(A|XQ_E)$ and others.

# Results

- Applied our method to compute rates for DIRNG and DIQKD.



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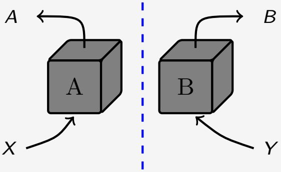
- Looked at different constraint sets $C$:
  - CHSH score

$$\frac{1}{4}\sum_{xy=a\oplus b} p(ab|xy) \geq \omega$$

  - Full distribution

$$p(ab|xy) = c_{abxy} \qquad \forall (a, b, x, y)$$



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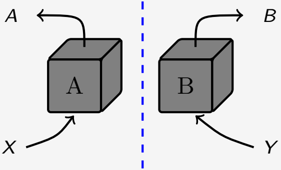


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- Investigated *detection efficiency* noise model.
  - Independent probability $\eta \in [0, 1]$ that each device *succeeds*.
  - Device failures recorded as a particular outcome.

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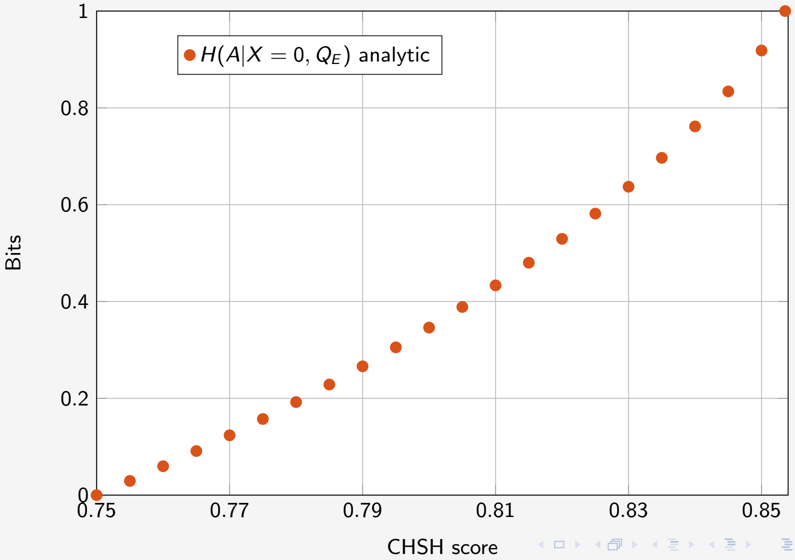
Bounding $\inf H(A|X=0, Q_E)$



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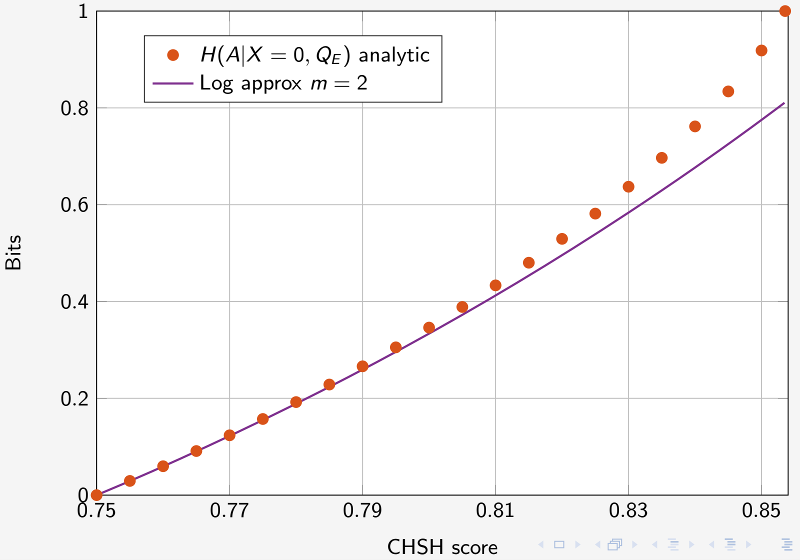

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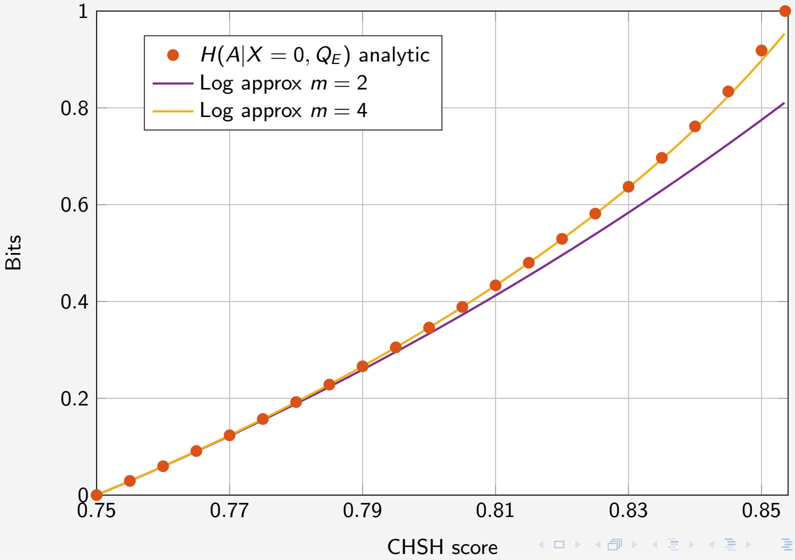

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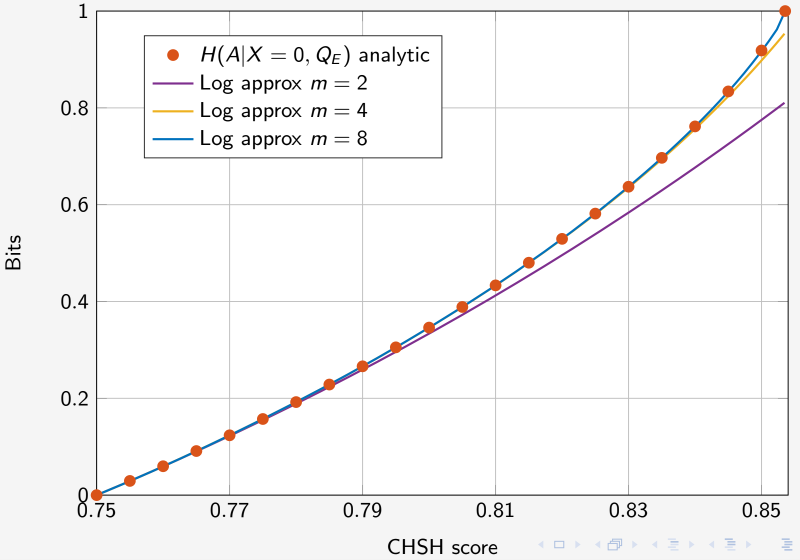
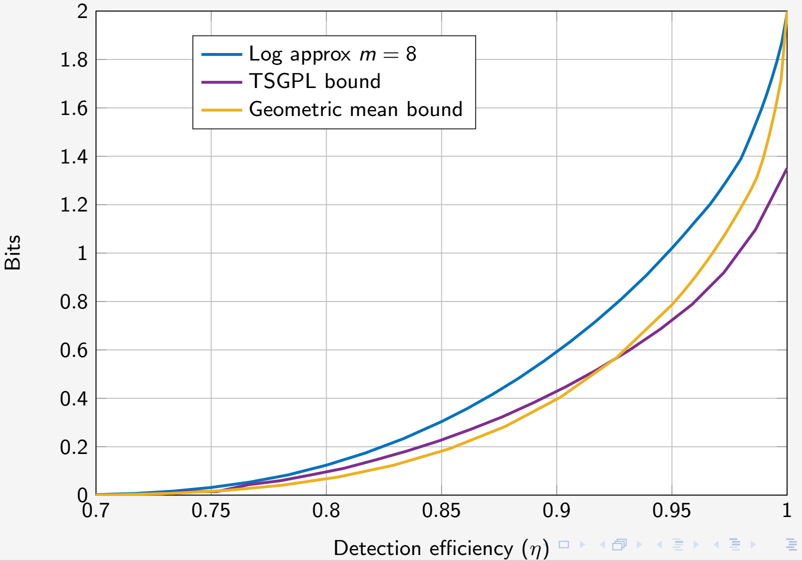


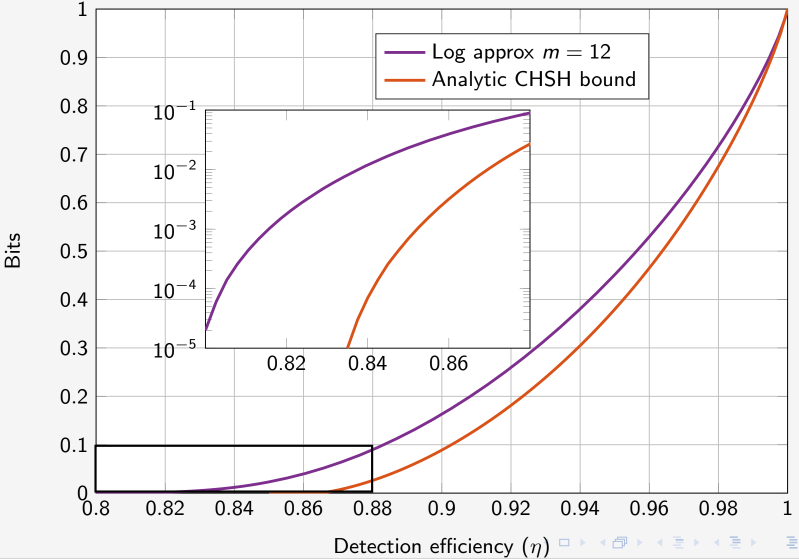
# Results II – Improved randomness expansion rates

Bounding $\inf H(AB|X = 0, Y = 0, Q_E)$



# Results III – Improved DIQKD rates

Bounding $\inf H(A|X=0,Q_E) - H(A|X=0,Y=2,B)$



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- Beyond DIQKD?

# Bibliography


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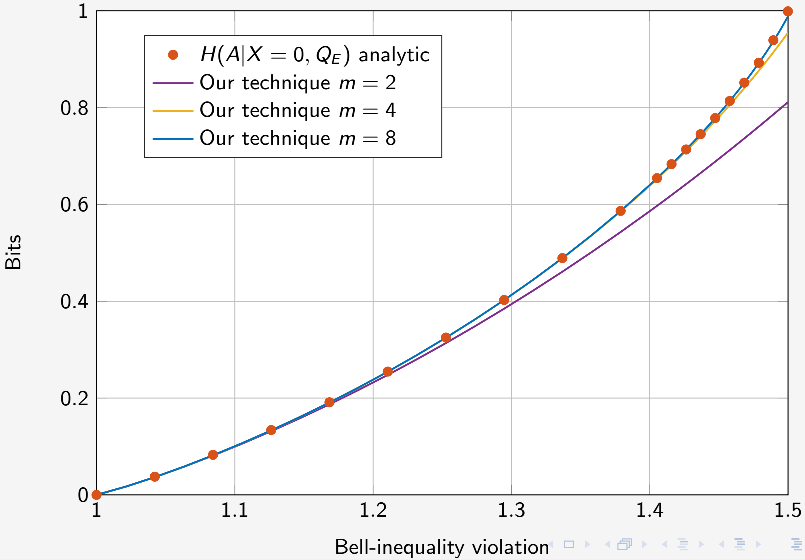
# Bonus results – DICKA setting (Holz inequality [HKB20])

Bounding inf $H(A|X = 0, Q_E)$



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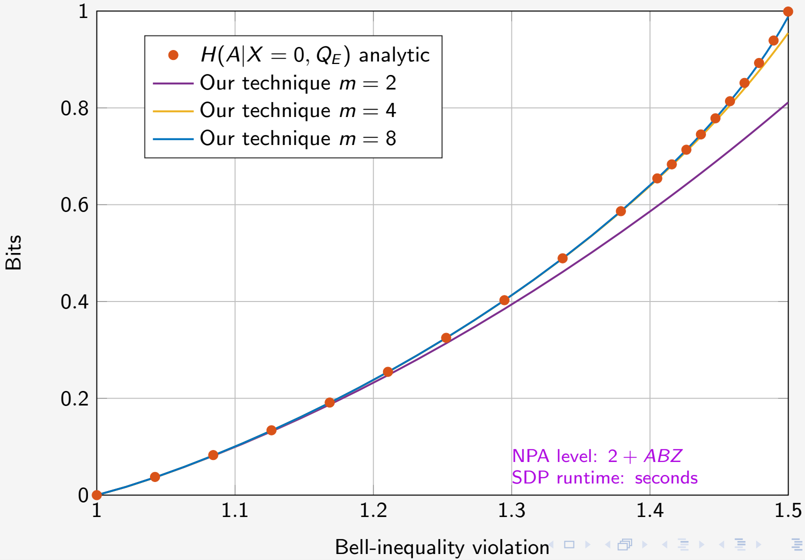
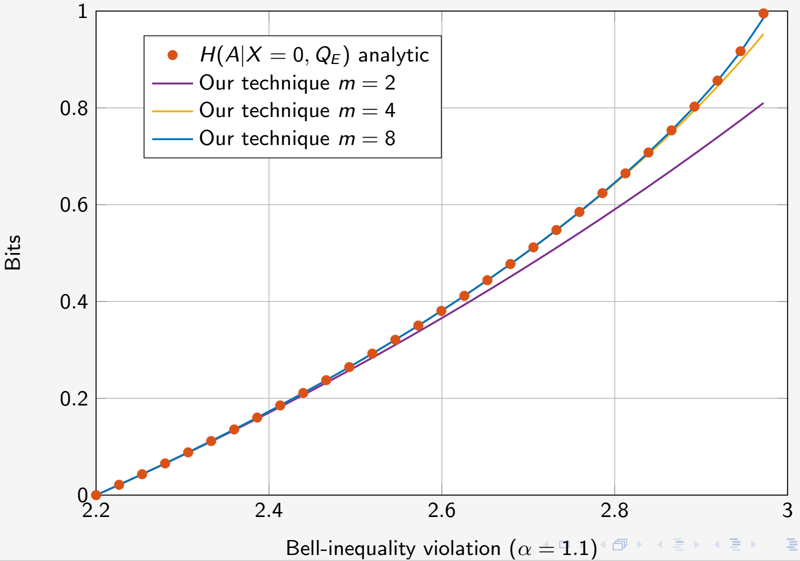


# Bonus results – Generalized CHSH [WAP21] ($\alpha = 1.1$)

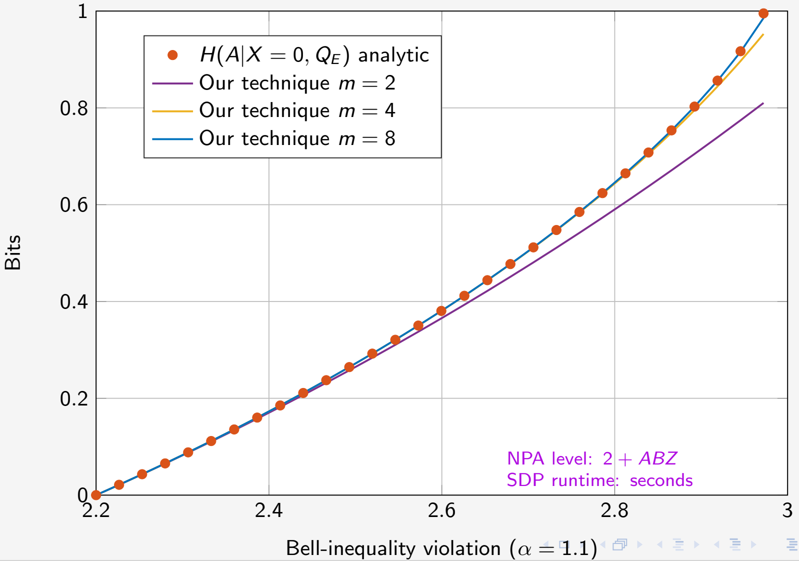
Bounding $\inf H(A|X = 0, Q_E)$

$$B_\alpha = \alpha(\langle A_0 B_0\rangle + \langle A_0 B_1\rangle) + \langle A_1 B_0\rangle - \langle A_1 B_1\rangle$$



# Bonus results – Generalized CHSH [WAP21] ($\alpha = 1.1$)

Bounding $\inf H(A|X=0, Q_E)$ $\qquad B_\alpha = \alpha(\langle A_0 B_0\rangle + \langle A_0 B_1\rangle) + \langle A_1 B_0\rangle - \langle A_1 B_1\rangle$



# Bonus results – Generalized CHSH [WAP21] ($\alpha = 0.9$)

Bounding $\inf H(A|X=0, Q_E)$ $\qquad$ $B_\alpha = \alpha(\langle A_0 B_0\rangle + \langle A_0 B_1\rangle) + \langle A_1 B_0\rangle - \langle A_1 B_1\rangle$